Quantum Stephani exact cosmological solutions and the selection of time variable

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Abstract

We study perfect fluid Stephani quantum cosmological model. In the present work the Schutz's variational formalism which recovers the notion of time is applied. This gives rise to Wheeler-DeWitt equation for the scale factor. We use the eigenfunctions in order to construct wave packets for each case. We study the time-dependent behavior of the expectation value of the scale factor, using many-worlds and deBroglie-Bohm interpretations of quantum mechanics.

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1 Introduction

In recent years observations show that the expansion of the Universe is accelerating in the present epoch [1] contrary to Friedmann-Robertson-Walker (FRW) cosmological models, with non-relativistic matter and radiation. Some different physical scenarios using exotic form of matter have been suggested to resolve this problem [2, 3, 4, 5, 6]. In fact the presence of exotic matter is not necessary to drive an accelerated expansion. Instead we can relax the assumption of the homogeneity of space, leaving the isotropy with respect to one point. The most general class of non-static, perfect fluid solutions of Einsteins equations that are conformally flat is known as the "Stephani Universe" [7, 8]. This model can be embedded in a five-dimensional flat pseudo-Euclidean space, which is not expansion-free and has non-vanishing density [9, 7, 10].

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In general, it has no symmetry at all, although its three dimensional spatial sections are homogeneous and isotropic [11]. The spherically symmetric Stephani Universes and some of their subcases have been examined in numerous papers [8]. So it may be important to study the quantum behavior of this models.

The notion of time can be recovered in some cases of quantum cosmology, for example when gravity is coupled to a perfect fluid [12, 13, 14]. This kind of systems are often studied as follows [16, 13, 14]. First one uses the Schutzs formalism for the description of the perfect fluid [17, 18], second one selects the dynamical variable of perfect fluid as the reference time. Finally, one uses canonical quantization to obtain the Wheeler-Dewitt (WD) equation in minisuperspace, which is a Schrödinger-like equation [12]. After solving the equation, one can construct wave packets from the resulting modes. The wave packets can be used to compute the time-dependent behavior of the scale factor. If the selected time variable results in a close correspondence between the expectation value of the scale factor and the classical prediction (prediction of General Relativity) for long enough time, the selected time variable can be considered as acceptable. This approach has been extensively employed in the literature, indicating in general the suppression of the initial singularity [15, 16, 12, 19, 20, 13, 21, 14].

We can also study this situation from ontological interpretation of quantum mechanics [13, 22, 23, 24, 25, 26]. In this approach, the problem of time is solved in any situation (not only in the presence of the matter field). In fact, the ontological interpretation predicts that the system follows a real trajectory given by

$$p_q = S_{,q}, \tag{1}$$

where p_q is the momentum conjugated to the variable q. Here, S is the phase of the wave function ($\Psi = Re^{iS}$), where R and S are real functions. The equation of motion (1) is governed not only by a classical potential V but also by the following quantum potential $V_Q = \dfrac{\nabla^2 R}{R}$.

In the present paper, we use the formalism of quantum cosmology in order to quantize Stephani cosmological model in Schutz's formalism [17, 18] and find WD equation in minisu-

perspace. In the Schutz's variational formalism the wave function depends on the scale factor a and on the canonical variable associated to the fluid, which plays the role of time T. Here, we describe matter as a perfect fluid matter. Although, this is semiclassical from the start, but it has the advantage of defining a variable, connected with the matter degrees of freedom which can be identified with time and leads to a well-defined Hilbert space.

2 The Model

The action for gravity plus perfect fluid in Schutz's formalism is written as

$$S = \int_M d^4x\sqrt{-g}\, R + 2\int_{\partial M} d^3x\sqrt{h}\, h_{ab}\, K^{ab} + \int_M d^4x\sqrt{-g}\, p, \tag{2}$$

where K^{ab} is the extrinsic curvature and h_{ab} is the induced metric over the three-dimensional spatial hypersurface, which is the boundary ∂M of the four dimensional manifold M in units where $16\pi G = 1$ [27]. The last term of (2) represents the matter contribution to the total action, where p is the pressure. In Schutz's formalism [17, 18] the fluid's four-velocity is expressed in terms of five potentials ϵ, ζ, ξ, θ and τ:

$$u_\nu = \frac{1}{\mu}(\epsilon_{,\nu} + \zeta\xi_{,\nu} + \theta\tau_{,\nu}), \tag{3}$$

where μ is the specific enthalpy, the variable τ is the specific entropy, while the potentials ζ and ξ are connected with rotation and are absent in models of the FRW type. The variables ϵ and θ have no clear physical meaning. The four-velocity satisfies the following normalization condition

$$u^\nu u_\nu = -1. \tag{4}$$

The metric in spherically symmetric Stephani Universe [9, 28, 7, 11, 8, 29] has the following form,

$$ds^2 = -\left[F(t)\frac{a(t)}{V(r,t)}\frac{d}{dt}\left(\frac{V(r,t)}{a(r,t)}\right)\right]^2 dt^2 + \frac{a^2(t)}{V^2(r,t)}\left(dr^2 + r^2 d\Omega^2\right), \tag{5}$$

where the functions $V(r,t)$ and $F(t)$ are defined as

$$V(r,t) = 1 + \frac{1}{4}k(t)r^2, \tag{6}$$

$$F(t) = \frac{a(t)}{\sqrt{C^2(t)a^2(t) - k(t)}}. \tag{7}$$

Using the line element (5) and the Einstein's equation, one can easily show the functions $C(t)$, $k(t)$ and $a(t)$ are not all independent but are related to each other with the expressions

$$\rho(t) = \frac{3C^2(t)}{8\pi G}, \tag{8}$$

$$p(r,t) = \frac{1}{8\pi G}[2C(t)\dot{C}(t)\frac{V(r,t)/a(t)}{(V(r,t)/a(t))^{\dot{}}} - 3C^2(t)], \tag{9}$$

where an overdot denotes a derivative with respect to t. Note that in the spherically symmetric Stephani models with the given coordinate system, the energy density $\rho(t)$ is uniform, contrary to the pressure $p(r,t)$, which is not uniform and depends on the distance from the symmetry center at $r = 0$. This is the reason for the absence of the barotropic equation of state ($p = p(\rho)$) in such models. However, if we assume some relations between $\rho(t)$ and $p(r,t)$, this could allow us to eliminate one of the unknown functions such as $C(t)$. Therefore we are left with two unknown functions $k(t)$ and $a(t)$. The first one $k(t)$, plays the role of a spatial curvature, and the second one $a(t)$, is the Stephani version of the FRW scale factor.

Now, we consider an observer which is placed at the symmetry center of the spherically symmetric Stephani Universe and our physical assumptions will concern in the vicinity of $r \approx 0$. First, we assume that locally, matter in the Universe satisfies the barotropic equation of state

$$p(r \approx 0, t) = \alpha\rho(t). \tag{10}$$

By substituting the Stephani metric (Eq. (5)) in the action (Eq. (2)) and Choosing a curvature function $k(t)$ in the form [30]

$$k(t) = \beta a^{\gamma}(t), \tag{11}$$

and after some thermodynamical considerations [12], the final reduced effective action near

$r \approx 0$, takes the form

$$S = \int dt \Big[-6\frac{\dot{a}^2 a}{N} + 6\beta N a^{1+\gamma} + N^{-1/\alpha} a^3 \frac{\alpha}{(\alpha+1)^{1/\alpha+1}} (\dot{\epsilon} + \theta\dot{\tau})^{1/\alpha+1} \exp\Big(-\frac{\tau}{\alpha}\Big)\Big]. \tag{12}$$

The reduced action may be further simplified by canonical methods [12] to the super-Hamiltonian

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6\beta a^{1+\gamma} + \frac{p_\epsilon^{\alpha+1} e^\tau}{a^{3\alpha}}, \tag{13}$$

where $p_a = -12\dot{a}a/N$ and $p_\epsilon = -\rho_0 u^0 N a^3$, ρ_0 being the rest mass density of the fluid. Using the canonical transformations

$$\begin{aligned} T &= p_\tau e^{-\tau} p_\epsilon^{-(\alpha+1)}, \qquad && p_T = p_\epsilon^{\alpha+1} e^\tau, \\ \bar{\epsilon} &= \epsilon - (\alpha+1)\frac{p_\tau}{p_\epsilon}, \qquad && \bar{p}_\epsilon = p_\epsilon, \end{aligned} \tag{14}$$

which are the generalization of the ones used in Ref. [12], the super-Hamiltonian takes the form

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6\beta a^{1+\gamma} + \frac{p_T}{a^{3\alpha}}\ , \tag{15}$$

where the momentum p_T is the only remaining canonical variable associated with matter which appears linearly in the super-Hamiltonian.

The classical dynamics is governed by the Hamilton equations, derived from Eq. (15) and Poisson brackets, namely

$$\begin{cases} \dot{a} = & \{a, N\mathcal{H}\} = -\dfrac{Np_a}{12a}\,, \\ \dot{p}_a = & \{p_a, N\mathcal{H}\} = -\dfrac{Np_a^2}{24a^2} + 6N(1+\gamma)\beta a^\gamma + \dfrac{3N\alpha\, p_T}{a^{1+3\alpha}}\,, \\ \dot{T} = & \{T, N\mathcal{H}\} = Na^{-3\alpha}\,, \\ \dot{p}_T = & \{p_T, N\mathcal{H}\} = 0\,. \end{cases} \tag{16}$$

We also have the constraint equation $\mathcal{H} = 0$. Choosing the gauge $N = a^{3\alpha}$, we have the following equations for the system

$$T = t, \tag{17}$$

$$\begin{aligned} \ddot{a} &= (3\alpha - \frac{1}{2})\frac{\dot{a}^2}{a} - \frac{1}{2}(1+\gamma)\beta a^{6\alpha+\gamma-1} - \frac{\alpha}{4}a^{3\alpha-2}p_T, && (18)\\ 0 &= -\frac{6\dot{a}^2}{a^{6\alpha-1}} - 6\beta a^{\gamma+1} + \frac{p_T}{a^{3\alpha}}. && (19) \end{aligned}$$

Note that the classical equations for $\gamma = +1$ case in Ref. [31], are corresponding with choosing the gauge $N = 1$. In this gauge the constraint equation $\mathcal{H} = 0$ reduces to

$$-6a\dot{a}^2 - 6\beta a^{\gamma+1} + a^{-3\alpha}p_T = 0, \tag{20}$$

or

$$\left(\frac{da(t)}{dt}\right)^2 + \beta a(t) = \frac{p_T}{6a^{3\alpha+1}(t)}. \tag{21}$$

Imposing the standard quantization conditions on the canonical variables ($p_a \to -i\frac{\partial}{\partial a}$, $p_T \to -i\frac{\partial}{\partial T}$) and demanding that the super-Hamiltonian operator (15) annihilate the wave function, we are led to the following WD equation in minisuperspace ($\hbar = 1$)

$$\frac{\partial^2\Psi}{\partial a^2} - 144\beta a^{2+\gamma}\Psi - i24a^{1-3\alpha}\frac{\partial\Psi}{\partial t} = 0. \tag{22}$$

According to the equation (17) $T = t$ can be associated with the time coordinate [32, 33]. Equation (22) takes the form of a Schrödinger equation $i\partial\Psi/\partial t = \hat{H}\Psi$. As discussed in [20, 33], in order for the Hamiltonian operator $\hat{H}$ to be self-adjoint the inner product of any two wave functions Φ and Ψ must take the form

$$(\Phi, \Psi) = \int_0^\infty a^{1-3\alpha}\Phi^*\Psi da. \tag{23}$$

On the other hand, the wave functions should satisfy the following boundary conditions [33, 34]

$$\Psi(0,t) = 0 \quad \text{or} \quad \left.\frac{\partial\Psi(a,t)}{\partial a}\right|_{a=0} = 0. \tag{24}$$

The WD equation (22) can be solved by separation of variables as follows,

$$\Psi(a,t) = e^{iEt}\psi(a), \tag{25}$$

where the scale factor dependent part of the wave function ($\psi(a)$) satisfies

$$-\psi'' + 144\beta a^{2+\gamma}\psi = 24Ea^{1-3\alpha}\psi, \tag{26}$$

and the prime denotes derivative with respect to a.

An interesting feature of the Stephani model is that the spatial curvature is time-dependent. The recent observational data show that our Universe is spatially flat. Moreover, negative powers in equation (11) lead to the spatially flat Universe in the present epoch. In the next section we solve a general class of exactly solvable models which $\gamma = -(1+3\alpha)$ and find the corresponding eigenfunctions. Then we construct the wave packets by appropriate superimposing of these eigenfunctions and compute the expectation value of the scale factor versus time. After solving the classical equations exactly, we compare the classical and quantum solutions and show that these solutions are asymptotically the same. Moreover, we study the case using de-Broglie Bohm interpretation of quantum mechanics and find the corresponding bohmian trajectories which are different from the classical case for small times due to the effect of the quantum potential.

3 Quantum cosmological models with $\gamma = -(1+3\alpha)$

In this Section, we consider the particular relation between γ and α as $\gamma = -(1+3\alpha)$. In this case the WD equation (26) takes the form

$$\psi'' + 24(E - 6\beta)a^{1-3\alpha}\psi = 0, \tag{27}$$

and hence equation (25) has the following general time-dependent solutions under the form of Bessel functions

$$\Psi_E(a,t) = e^{i(E-6\beta)t}\sqrt{a}\Big[c_1 J_{\frac{1}{3(1-\alpha)}}\Big(\frac{\sqrt{96(E-6\beta)}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\Big)+c_2 Y_{\frac{1}{3(1-\alpha)}}\Big(\frac{\sqrt{96(E-6\beta)}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\Big)\Big]. \tag{28}$$

Wave packets can be constructed by superimposing these solutions to obtain physically allowed wave functions. The general structure of these wave packets are

$$\Psi(a,t)=\int_{6\beta}^{\infty}A(E)\Psi_E(a,t)dE. \tag{29}$$

We choose $c_2=0$ to satisfy the first boundary condition of Eq. (24). Defining $r=\frac{\sqrt{96(E-6\beta)}}{3(1-\alpha)}$, simple analytical expressions for the wave packet can be found by choosing $A(E)$ to be a quasi-gaussian function

$$\Psi(a,t)=\sqrt{a}\int_0^{\infty}r^{\nu+1}e^{-\kappa r^2+i(\frac{3}{32}r^2(1-\alpha)^2+6\beta)t}J_\nu(ra^{\frac{3(1-\alpha)}{2}})dr, \tag{30}$$

where $\nu=\frac{1}{3(1-\alpha)}$ and κ is an arbitrary positive constant. The above integral is known [35], and the wave packet takes the form

$$\Psi(a,t)=a\frac{e^{-\frac{a^{3(1-\alpha)}}{4B}+6i\beta t}}{(-2B)^{\frac{4-3\alpha}{3(1-\alpha)}}}, \tag{31}$$

where $B=\kappa-i\frac{3}{32}(1-\alpha)^2t$. Now, we can verify what these quantum models predict for the behavior of the scale factor of the Universe. By adopting the many-worlds interpretation [36], and with regards to the inner product relation (23), the expectation value of the scale factor

$$\langle a\rangle_t=\frac{\int_0^{\infty}a^{1-3\alpha}\Psi(a,t)^*a\Psi(a,t)da}{\int_0^{\infty}a^{1-3\alpha}\Psi(a,t)^*\Psi(a,t)da}, \tag{32}$$

is easily computed, leading to

$$\langle a\rangle_t=\frac{\Gamma\left(\frac{3\alpha-5}{3(\alpha-1)}\right)}{\Gamma\left(\frac{3\alpha-4}{3(\alpha-1)}\right)}\left[\frac{\frac{18(1-\alpha)^4}{(32)^2}t^2+2\kappa^2}{\kappa}\right]^{\frac{1}{3(1-\alpha)}}. \tag{33}$$

Now, we can calculate the dispersion of the wave packets

$$(\Delta a)_t^2=\langle a^2\rangle_t-\langle a\rangle_t^2, \tag{34}$$

using (31,32), we have

$$(\Delta a)_t^2=\frac{3\pi\Gamma\left(\frac{1}{1-\alpha}\right)-16^{\frac{1}{3-3\alpha}}\Gamma\left(\frac{1}{3-3\alpha}\right)\Gamma\left(\frac{5-3\alpha}{6-6\alpha}\right)^2}{\pi\Gamma\left(\frac{1}{3-3\alpha}\right)}\left[\frac{\frac{18(1-\alpha)^4}{(32)^2}t^2+2\kappa^2}{\kappa}\right]^{\frac{2}{3(1-\alpha)}}. \tag{35}$$

This shows the dispersion of the wave packets through the time with the minimum at $t = 0$. This is similar to the free particle case, where the the wave packets disperse more rapidly for more localized initial states.

The important feature of this model is the avoidance of the singularity. Equation (33) shows that the expectation value of the scale factor never vanishes for all time. On the other hand, at the quantum level, since the probability density of finding the scale factor at $a = 0$ (with regards to the inner product relation (23) and the behavior of Bessel functions for small values of the argument) is zero in all times ($\lim_{a\to 0} a^{1-3\alpha}|\Psi(a,t)|^2 = 0$), we have a indication that these models may not have singularities at the quantum level.

In classical case, by eliminating the p_T variable in the equations of motions (18,19), the resulting equation in case $\gamma = -(1+3\alpha)$ takes the following simple form

$$\ddot{a} = (3\alpha - 1)\frac{\dot{a}^2}{2a}, \tag{36}$$

which has the exact solution as

$$a(t) = a_0 t^{\frac{2}{3(1-\alpha)}}, \tag{37}$$

where

$$a_0 = \frac{\Gamma\left(\frac{3\alpha-5}{3(\alpha-1)}\right)}{\Gamma\left(\frac{3\alpha-4}{3(\alpha-1)}\right)}\left[\frac{18(1-\alpha)^4}{(32)^2\kappa}\right]^{\frac{1}{3(1-\alpha)}}. \tag{38}$$

Figure 1 shows the behavior of the classical scale factor (37) and quantum mechanical expectation value of the scale factor (37) versus time for various cases.

It is known that the results obtained by using the many-worlds interpretation agree with those that can be obtained using the ontological interpretation of quantum mechanics [22, 23, 24, 25, 26]. In Bohmian interpretation the wave function is written as

$$\Psi = R\, e^{iS} \tag{39}$$

where R and S are real functions. Inserting this expression in the WD equation (22) (for $\gamma = -(1+3\alpha)$), we have

$$\frac{\partial S}{\partial t} - \frac{1}{24a^{1-3\alpha}}\left(\frac{\partial S}{\partial a}\right)^2 - 6\,\beta + Q \;=\; 0, \tag{40}$$

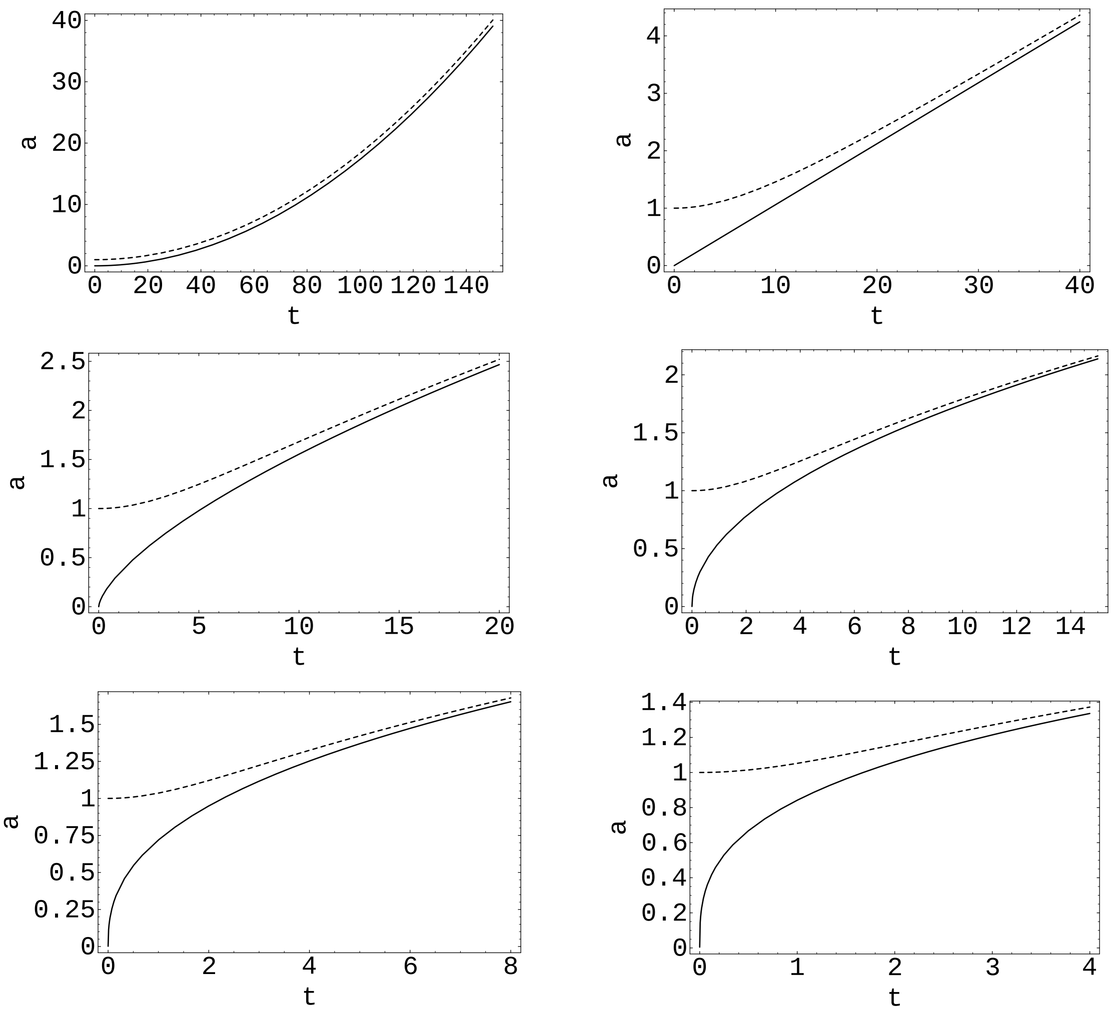

Figure 1: Classical scale factor and the expectation value of the scale factor versus time for $\alpha = 2/3$ (up, left), $\alpha = 1/3$ (up, right), $\alpha = 0$ (middle, left), $\alpha = -1/3$ (middle, right), $\alpha = -2/3$ (down, left), and $\alpha = -1$ (down, right).

$$\frac{\partial R}{\partial t} - \frac{1}{12a^{1-3\alpha}}\frac{\partial R}{\partial a}\frac{\partial S}{\partial a} - \frac{1}{24a^{1-3\alpha}}R\frac{\partial^2 S}{\partial a^2} \quad = \quad 0. \tag{41}$$

Here, $Q = \frac{1}{24a^{1-3\alpha}}\frac{1}{R}\frac{\partial^2 R}{\partial a^2}$ is the quantum potential which modifies the Hamilton-Jacobi equation. In fact, the deviation from the classical trajectory happens whenever the quantum potential is more important than the classical potential. The real functions $R(a,t)$ and $S(a,t)$ can be obtained from the wave function (31) as

$$R = \left[4\kappa^2 + \left(\frac{3}{16}\right)^2 (1-\alpha)^4 t^2\right]^{-\frac{4-3\alpha}{6(1-\alpha)}} a \exp\left\{-\frac{\kappa a^{3(1-\alpha)}}{4\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]}\right\}, \tag{42}$$

$$S = -\frac{3}{128}\frac{(1-\alpha)^2 a^{3(1-\alpha)} t}{\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]} + \frac{(4-3\alpha)}{3(1-\alpha)} \arctan\left[\frac{3}{32}\frac{(1-\alpha)^2 t}{\kappa}\right] + 6\beta t. \tag{43}$$

In the Bohmian interpretation, the behavior of the scale factor is governed by the following equation

$$p_a = \frac{\partial S}{\partial a}. \tag{44}$$

On the other hand, from the definition of p_a (16) and for $N = a^{3\alpha}$ we have $p_a = -12a^{1-3\alpha}\dot{a}$. Therefore, Bohmian trajectory becomes

$$512\frac{\dot{a}}{a} = 3(1-\alpha)^3 \frac{t}{\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]}, \tag{45}$$

which the integration yields

$$a(t) = a_0\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]^{\frac{1}{3(1-\alpha)}}, \tag{46}$$

where a_0 is the constant of the integration. This completely coincides with the computation of the expectation value of the scale factor. Now we can find the quantum potential

$$Q(a,t) = -\frac{\kappa}{32}\frac{1-\alpha}{\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]^2}\left\{3\kappa(1-\alpha)a^{3(1-\alpha)} - (4-3\alpha)\left[\kappa^2 + \left(\frac{3}{32}\right)^2 (1-\alpha)^4 t^2\right]\right\}. \tag{47}$$

Using the relation between the scale factor and time (46) the quantum potential can be written in terms of the scale factor as

$$Q(a) = \kappa\frac{1-\alpha}{32} a_0^{3(1-\alpha)} \frac{(4-3\alpha) - 3\kappa(1-\alpha)a_0^{3(1-\alpha)}}{a^{3(1-\alpha)}}. \tag{48}$$

It is obvious that the quantum effects are negligible for large values of the scale factor and are important for small values of the scale factor. Therefore, asymptotically we have the classical behavior.

In the next section for completeness we consider briefly some interesting and exactly solvable cases which have bound state solutions.

4 Bound state solutions

In this Section, we study four different cases of γ and α. We find the exact discrete energy spectrum and corresponding eigenfunctions.

For $\gamma = -1$ and $\alpha = 1/3$ (radiation), the WD equation (26) reduces to

$$-\psi'' + 144\beta a\psi = 24E\psi, \tag{49}$$

which can be rewritten as

$$\psi'' - 144\beta\left(a - \frac{E}{6\beta}\right)\psi = 0, \tag{50}$$

by taking $x = a - \frac{E}{6\beta}$ we have

$$\frac{d^2}{dx^2}\psi(x) - 144\beta x\psi(x) = 0, \tag{51}$$

which is the Airy's differential equation [37]. This equation has two solutions as $\mathrm{Ai}\left[(144\beta)^{1/3}x\right]$ and $\mathrm{Bi}\left[(144\beta)^{1/3}x\right]$. First one is exponentially decreasing function of x and the second one grows exponentially and is physically unacceptable. Therefore, the solution is

$$\psi(a) = \mathrm{Ai}\left[(144\beta)^{1/3}\left(a - \frac{E}{6\beta}\right)\right]. \tag{52}$$

We choose the first boundary condition (24), which leads to

$$\mathrm{Ai}\left[-E\left(\sqrt{\frac{3}{2}}\beta\right)^{-2/3}\right] = 0. \tag{53}$$

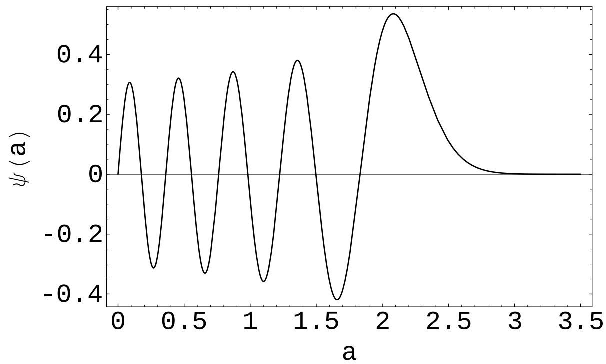

Figure 2: Plot of the wave function ($\psi(a)$) for $\beta = 1$ and $n = 8$, showing the oscillatory behavior for the small values of the scale factor and exponential damping for the large values of the scale factor.

Airy's function $\mathrm{Ai}(x)$ has infinitely many negative zeros $z_n = -a_n$, where $a_n > 0$, therefore, the energy levels quantize and take the values

$$E_n = \left(\sqrt{\frac{3}{2}}\beta\right)^{2/3} a_n. \tag{54}$$

The time-dependent eigenfunctions take the form

$$\Psi_n(a,t) = e^{iE_n t}\mathrm{Ai}\left[(144\beta)^{1/3}\left(a - \frac{E_n}{6\beta}\right)\right]. \tag{55}$$

It is important to note that Airy's function $\mathrm{Ai}(x)$ has an oscillatory behavior for $x < 0$ ($a < \frac{E_n}{6\beta}$) whiles for $x > 0$ ($a > \frac{E_n}{6\beta}$) decreases monotonically and is an exponentially damped function for large x (Fig. 2). Therefore, the solutions (55) show a classical behavior for small a and a quantum behavior for large a. This is contrary to usually expected results in the previous Section. In fact detecting quantum gravitational effects in large Universes is noticeable which has been also observed in FRW and Kaluza-Klein models [32, 38].

In $\gamma = -1$ and $\alpha = -1/3$ (cosmic strings) case, the WD equation (26) reduces to

$$-\psi'' + 144\beta a\psi = 24Ea^2\psi. \tag{56}$$

For $E < 0$ the above equation can be written as

$$-\psi'' + 24|E|\left[\left(a - \frac{3\beta}{E}\right)^2 - \left(\frac{3\beta}{E}\right)^2\right]\psi = 0, \tag{57}$$

by taking $x = a - \dfrac{3\beta}{E}$ we have

$$-\frac{d^2}{dx^2}\psi(x) + 24|E|x^2\psi(x) = \frac{216\beta^2}{|E|}\psi(x). \tag{58}$$

This equation is identical to the time-independent Schrödinger equation for a harmonic oscillator with unit mass and energy λ:

$$-\frac{d^2\psi(x)}{dx^2} + \omega^2 x^2\psi(x) = 2\lambda\psi(x), \tag{59}$$

where $2\lambda = \dfrac{216\beta^2}{|E|}$ and $\omega^2 = 24|E|$. Therefore, the allowed values of λ are $\omega(n+1/2)$ and the possible values of E are

$$E_n = -\left(\frac{113\beta^2}{\sqrt{24}(n+\frac{1}{2})}\right)^{\frac{2}{3}} \quad , \qquad n = 0, 1, 2, ... \quad . \tag{60}$$

therefore, the stationary solutions are

$$\Psi_n(a,t) = e^{iE_nt}\varphi_n\left(a - \frac{3\beta}{E_n}\right), \tag{61}$$

$$\varphi_n(x) = H_n\left((24|E|)^{\frac{1}{4}}x\right)e^{-\sqrt{6|E|}\,x^2}, \tag{62}$$

where H_n are Hermite polynomials. The wave functions (61) are similar to the stationary quantum wormholes as defined in [39]. However, neither of the boundary conditions (24) can be satisfied by these wave functions.

For $\gamma = -2$ and $\alpha = 0$ (dust regime), the WD equation (26) reduces to

$$-\psi'' + 144\beta\psi = 24Ea\psi, \tag{63}$$

which can be written as

$$\psi'' - 24E\left(\frac{6\beta}{E} - a\right)\psi = 0, \tag{64}$$

by taking $x = \dfrac{6\beta}{E} - a$ we have

$$\psi'' - 24Ex\psi = 0, \tag{65}$$

which is again the Airy's differential equation [37]. Therefore, the physically acceptable solution is

$$\psi(a) = \mathrm{Ai}\left[(24E)^{1/3}\left(\frac{6\beta}{E} - a\right)\right]. \tag{66}$$

We choose the first boundary condition (24), which leads to

$$\mathrm{Ai}\left[12\sqrt[3]{2}\beta E^{-2/3}\right] = 0. \tag{67}$$

Airy's function $\mathrm{Ai}(x)$ has infinitely many negative zeros z_n, therefore, the energy levels quantize and take the values of

$$E_n = \left(\frac{12\sqrt[3]{2}\beta}{z_n}\right)^{3/2}, \tag{68}$$

which exist only for negative values of β. The time-dependent eigenfunctions take the form

$$\Psi_n(a,t) = e^{iE_n t}\mathrm{Ai}\left[(24E_n)^{1/3}\left(\frac{6\beta}{E_n} - a\right)\right]. \tag{69}$$

These solutions (69) show a quantum behavior for small a and a classical behavior for large a.

In $\gamma = -2$ and $\alpha = -1/3$ (cosmic strings) case, the WD equation (26) reduces to

$$-\psi'' + 144\beta\psi = 24Ea^2\psi. \tag{70}$$

For $E < 0$ the above equation can be written as

$$-\psi'' + 24|E|a^2\psi = -144\beta\psi. \tag{71}$$

This equation is identical to the time-independent Schrödinger equation for a harmonic oscillator with unit mass and energy λ, where $2\lambda = -144\beta$ and $\omega^2 = 24|E|$. Therefore, the allowed values of λ are $\omega(n + 1/2)$ and the possible values of E are

$$E_n = -\frac{216\beta^2}{(n+\frac{1}{2})^2}, \qquad n = 0, 1, 2, ... \quad , \tag{72}$$

for $\beta < 0$. Thus the stationary solutions are

$$\Psi_n(a,t) = e^{iE_n t}\varphi_n(a), \tag{73}$$

$$\varphi_n(a) = H_n\left((24|E_n|)^{\frac{1}{4}}a\right)e^{-\sqrt{6|E_n|}\,a^2}. \tag{74}$$

The solutions for odd n satisfy the first boundary condition (24) and the appropriate wave packets can be constructed by superposing these stationary solutions.

5 Conclusion

In this work we have investigated perfect fluid Stephani quantum cosmological models. The use of Schutz's formalism allowed us to obtain a Schrödinger-like WD equation in which the only remaining matter degree of freedom plays the role of time. We have obtained eigenfunctions and therefore constructed the acceptable wave packets by appropriate linear combination of these eigenfunctions. The time evolution of the expectation value of the scale factor has been determined using the many-worlds and Bohmian interpretations of quantum cosmology. We have shown that contrary to the classical case, the expectation values of the scale factor avoid singularity in the quantum case. At the end, we solved some interesting bound state cases and found their discrete energy eigenvalues. We have also shown that in some bound state cases, we may observe the quantum effects in the large scales which correspond to the quantum behavior at the late time cosmology.

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